PROGRESIONES GEOMÉTRICAS

Problema 36:

Encontrar el quinto término a_5 de la progresión geométrica cuyos dos primeros términos son:

$$a_1 = 3$$

$$a_2 = \frac{3}{\sqrt{5}-1}$$

Hallar la suma de la serie geométrica:

$$a_1 + a_2 + \cdots = 3 + \frac{3}{\sqrt{5}-1} + \cdots$$

Solución Problema 36:

Sabemos que la razón de una progresión geométrica:

$$r = \frac{a_2}{a_1}$$

$$r = \frac{\frac{3}{\sqrt{5}-1}}{3} = \frac{3}{3(\sqrt{5}-1)} = \frac{1}{(\sqrt{5}-1)}$$

A continuación racionalizamos el denominador:

$$r = \frac{1}{(\sqrt{5}-1)} = \frac{(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)} = \frac{(\sqrt{5}+1)}{5-1} = \frac{(\sqrt{5}+1)}{4}$$

Sabemos que:

$$a_n = a_1 r^{\mathrm{n}-1}$$

$$a_5 = 3\left(\frac{\sqrt{5}+1}{4}\right)^4 = 3\left(\frac{\sqrt{5}}{4} + \frac{1}{4}\right)^4$$

Para mayor claridad desarrollamos el binomio de Newton aparte:

$$\binom{m}{n} = \frac{m!}{n!.\,(m-n)!}$$

Sabemos que:

$$\binom{4}{0} = \binom{4}{4} = \frac{4!}{0!.4!} = \frac{4!}{4!.0!}\ 1$$

$$\binom{4}{1} = \binom{4}{3} = \frac{4!}{1!.3!} = \frac{4!}{3!.1!} = 4$$

$$\binom{4}{2} = \frac{4!}{2!.2!} = 6$$

Luego:

$$(\frac{\sqrt{5}}{4} + \frac{1}{4})^4 = \binom{4}{0}(\frac{\sqrt{5}}{4})^4 + \binom{4}{1}(\frac{\sqrt{5}}{4})^3\frac{1}{4} + \binom{4}{2}\left(\frac{\sqrt{5}}{4}\right)^2(\frac{1}{4})^2 + \binom{4}{3}\frac{\sqrt{5}}{4}(\frac{1}{4})^3 + \binom{4}{4}(\frac{1}{4})^4$$

$$= (\frac{\sqrt{5}}{4})^4 + 4(\frac{\sqrt{5}}{4})^3\frac{1}{4} + 6\left(\frac{\sqrt{5}}{4}\right)^2(\frac{1}{4})^2 + 4\frac{\sqrt{5}}{4}(\frac{1}{4})^3 + (\frac{1}{4})^4 =$$

$$= \frac{25}{256} + \frac{5\sqrt{5}}{64} + 6\frac{5}{16}.\frac{1}{16} + \frac{\sqrt{5}}{64} + \frac{1}{256} =$$

$$= \frac{25}{256} + \frac{5\sqrt{5}}{64} + \frac{30}{256} + \frac{\sqrt{5}}{64} + \frac{1}{256} = \frac{25 + 20\sqrt{5} + 30 + 4\sqrt{5} + 1}{256} = \frac{56 + 24\sqrt{5}}{256}$$

$$= \frac{7 + 3\sqrt{5}}{32}$$

Una vez hallado, tenemos que:

$$(\frac{\sqrt{5}}{4} + \frac{1}{4})^4 = \frac{7 + 3\sqrt{5}}{32}$$

Luego a_5 será:

$$a_5 = 3(\frac{\sqrt{5}+1}{4})^4 = 3(\frac{\sqrt{5}}{4} + \frac{1}{4})^4 = 3.\frac{7 + 3\sqrt{5}}{32} = \frac{21 + 9\sqrt{5}}{32}$$

Hallar la suma de la serie geométrica:

$$a_1 + a_2 + \cdots = 3 + \frac{3}{\sqrt{5} - 1} + \cdots$$

Sabemos la fórmula de la suma de la progresión geométrica ilimitada:

$$S_n = \frac{a_1}{1 - r}$$

$$S_n = \frac{3}{1 - \frac{(\sqrt{5}+1)}{4}} = \frac{3}{\frac{4-(\sqrt{5}+1)}{4}} = \frac{3.4}{4-(\sqrt{5}+1)} = \frac{12}{4-\sqrt{5}-1} =$$

$$\frac{12}{3-\sqrt{5}} = \frac{12(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})} = \frac{12(3+\sqrt{5})}{9-5} = \frac{12(3+\sqrt{5})}{4} = 3(3+\sqrt{5})$$

$$S_n = 9 + 3\sqrt{5}$$